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AN ALTERNATIVE FOR THE GAME PROBLEM OF CONVERGENCE

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In this paper a new class of generalized mixed strategies of players is presented, related to the problem of bringing a motion, under a control involving conflict, to a specified set under a phase restriction. This class of problems is so wide that it includes strategies which give saddle-point type situations in typical differential games. The contents of this paper are related to the problems discussed in [1–4] and the discussions are based on the extremal construction introduced in [5–7].

1. Consider first a motion under control involving conflict described by

$$dx/dt = f(t, x, u, v), \quad x[t_0] = x_0 \tag{1.1}$$

where x is the n-dimensional phase vector of the system, u and v are the control force vectors of the first and the second player, respectively, and $f(t, x, u, v)$ is a continuous vector function satisfying the Lipshits condition in x. It is assumed that the players can only choose the controls u and v restricted by the condition

$$u \in P, \qquad v \in Q \tag{1.2}$$

where P and Q are closed and bounded sets in the corresponding vector spaces.

In introducing the concept of mixed strategies of players and obtaining the motions of the system (1.1) generated by these strategies, the proposed definition of the strategies corresponds to the following information given to the players: – at any instant $t \geqslant t_0$ the players know the actual position of the game $p[t]=\{t, x[t]\}$ but have no knowledge of the control chosen by the partner at this and subsequent instants.

It is considered that this class of mixed strategies is sufficiently complete in the sense that for any initial time $\vartheta > t_0$ and initial position $p_0 = \{t_0, x_0\}$ of the game, the class contains either the strategy of the first player which guarantees that the motion (1.1) will converge to a given set at the instant $t = \vartheta$ and at the same time guarantees that a certain prescribed phase restriction will be fulfilled, or the strategy of the second player which ensures that all motions (1.1) bounded by the given phase restriction will evade the given set within the interval $[t_0, \vartheta]$. (This alternative is formulated more precisely at the end of this Section). In Sects. 2 and 3 below the description of the so-called extremal construction [5–7] is given, which is used to prove the validity of the alternative formulated in this Section.

This alternative enables the optimal strategies to be constructed which define saddle-point type situations in the differential games. In particular the findings of this paper may be used to study the following types of the game problems of dynamics:

1. A homing game problem with phase restrictions. Here a strategy must be constructed for the first player which brings the motion (1.1) to the given set in the shortest possible time ensuring that a certain phase restriction is fulfilled.

2. A differential game in which the motion is described by Eq. (1.1) and the payoff is given by

$$\gamma = \varphi(\vartheta, x[\vartheta]) + \int_{t_0}^{\vartheta} \psi(t, x[t])\, dt$$

where $\varphi(t, x)$ and $\psi(t, x)$ are given continuous functions and $\vartheta = \vartheta(x[\cdot])$ is the instant at which the point $p[t] = \{t, x[t]\}$ reaches some prescribed set N for the first time.

3. A differential game with the payoff in the following form:

$$\gamma = \max \varphi(t, x[t]) \text{ when } t_0 \leqslant t \leqslant \vartheta(x[\cdot])$$

where, as in the previous case, $\varphi(t, x)$ is a continuous function and $\vartheta(x[\cdot])$ is the instant at which the point $p[t] = \{t, x[t]\}$ reaches the prescribed set N.

The concepts of mixed strategies of the players may be introduced at this point allowing the motions of (1.1) corresponding to these strategies to be determined. Let $U = U(t, x)$ be a function defined for $t \geqslant t_0$ and all x which generated a one-to-one correspondence between the positions $p = \{t, x\}$ of the game and the set $U(t, x)$ of regular Borel measures $\mu(du)$ [8] normed on P, i.e. $\mu(P) = 1$.

Since $\mu(du)$ will be the only measures considered here, these will be called simply the measures $\mu(du)$ on P.

The function $U = U(t, x)$ specified above defines the mixed strategy of the first player and allows the motions of (1.1) generated by this strategy to be determined as follows.

Let Δ be some range set of nonoverlapping semiopen intervals

$$[\tau_i,\ \tau_{i+1}) \qquad (i=0,\ 1,\ 2,\ldots,\ \tau_0=t_*)$$

covering the semiopen line $[t_*,\ \infty)$. Then the approximate motion of the system (1.1) generated by the mixed strategy $U=U\ (t,\ x)$ of the first player, together with the trivial strategy V_τ of the second player, and corresponding to the range set Δ will be given by

$$x_\Delta\ [t]=x_\Delta\ [t;\ t_*,\ x_*\ U,\ V_\tau]$$

where $x_\Delta\ [t]$ is an absolutely continuous vector function satisfying the following recurrence relations:

$$\frac{dx_\Delta\,[t]}{dt}\in F\ (t,\ x_\Delta\,[t];\ \mu\,(du;\ p\ [\tau_i])) \tag{1.3}$$

$$x_\Delta\,[t_*]=x_\Delta\,[\tau_0]=x_*,\quad \tau_i\leqslant t<\tau_{i+1}\qquad (i=0,1,2,\ldots)$$

at nearly all $t\geqslant t_*$.

Here $x_\Delta\ [t_*]=x_*$ is the initial condition for the motion

$$x_\Delta\ [t]=x_\Delta\ [t;\ t_*,\ x_*,\ U,\ V_\tau]$$

and $F\ (t,\ x;\ \mu\ (du))$ is the convex hull of the set of all vectors of the form

$$f=\int f\ (t,\ x,\ u,\ v)\,\mu\ (du)$$

where $v\in Q$ and $\mu\ (du;\ p\ [\tau_i])$ is a measure on P belonging to the set $U\ (\tau_i,\ x_\Delta\ [\tau_i])$.

Let the concept of generalized motion of the system (1.1) now be introduced. Denote by $\sigma\ (\Delta)$ the quantity given by

$$\sigma\,(\Delta)=\sup\nolimits_i\,(\tau_{i+1}-\tau_i),\qquad i=0,1,2,\ \ldots \tag{1.4}$$

where τ_i are the boundary points of the semiopen intervals $[\tau_i,\ \tau_{i+1})$ of the range set Δ, and let the absolutely continuous vector function

$$x\ [t]=x\ [t;\ t_*,\ x_*,\ U,\ V_\tau]$$

describe a generalized motion of the system (1.1) satisfying the initial condition $x\ [t_*]=x_*$ generated by the mixed strategy $U=U\ (t,\ x)$ of the first player together with the trivial strategy V_τ of the second player, provided that the sequence of motions

$$x_{\Delta_k}\ [t]=x_{\Delta_k}\ [t;\ t_*,\ x_k,\ U,\ V_\tau]\qquad (k=1,2,\ldots)$$

exists such that the relation

$$\lim_{k\to\infty}\ x_{\Delta_k}\ [t;\ t_*,\ x_k,\ U,\ V_\tau]=x\ [t;\ t_*,\ x_*,\ U,\ V_\tau]$$

also holds on any finite segment $[t_*,\ t^*]$ uniformly and that the following conditions are fulfilled:

$$\lim\ \sigma\ (\Delta_k)=0,\qquad \lim\ x_k=x_*,\qquad k\to\infty$$

In the following discussion these generalized motions are simply called the motions of the system (1.1).

Some properties of the set of motions

$$x\ [t]=x\ [t;\ \ t_*,\ x_*,\ U,\ V_\tau]$$

may now be noted. Thus,

1° this set is nonempty, and

2° the set of motions $x\ [t;\ t_*,\ x_*,\ U,\ V_\tau]$ regarded as a collection of vector functions $x=x\ [t]$ defined on some finite interval $[t_*,\ t^*]$ is compact in itself

([9], p. 222) and depends on the initial condition x_* semicontinuously from above, relative to inclusion.

From the latter property the following condition holds: if $x_i \to x_*$ as $i \to \infty$, and the sequence of vector functions $x[t;\ t_*,\ x_i,\ U,\ V_\tau]$ converges uniformly on $[t_*,\ t^*]$ to some vector function $x_*[t]$, then on the segment $[t_*,\ t^*]$ the vector function $x_*[t]$ coincides with one of the motions $x[t;\ t_*,\ x_*,\ U,\ V_\tau]$.

In one particular case when the set $U(t,\ x)$ consists of a single element, a measure $\mu(du)$, and does not depend on the position $p = \{t, x\}$ of the game, the corresponding motions will be denoted by $x[t] = x[t;\ t_*,\ x_*,\ \mu(du),\ V_\tau]$.

In this case the set of motions $x[t;\ t_*,\ x_*,\ U,\ V_\tau]$ coincides with the set of solutions of the following differential equation in contingencies [10]:

$$\frac{dx[t]}{dt} \in F(t, x[t]; \mu(du)), \qquad x[t_*] = x_* \tag{1.5}$$

The concept of the mixed strategy of the second player is introduced in a similar manner. This is represented by a certain function $V \div V(t,\ x)$ (here $V(t, x)$ denotes the sets of regular measures $\nu(dv)$ normed on Q), and the motions of the system (1.1) corresponding to the strategy $V = V(t,\ x)$ and the trivial strategy of the first player U_τ, are defined by $x[t] = x[t;\ t_*,\ x_*,\ U_\tau,\ V]$. The properties of the set $x[t] = x[t;\ t_*,\ x_*,\ U_\tau,\ V]$ are the same as those of the set $x[t] = x[t;\ t_*,\ x_*, U,\ V_\tau]$.

The motions $x[t;\ t_*,\ x_*,\ U_\tau,\ V_\tau]$ corresponding to the pair of trivial strategies and the motions $x[t;\ t_*,\ x_*,\ U,\ V]$ generated by any two mixed strategies $U = U(t,\ x)$ and $V = V(t,\ x)$ will also be utilized in the discussion. In this case we shall regard as a motion $x[t;\ t_*,\ x_*,\ U_\tau,\ V_\tau]$ any continuous vector function $x[t]$ satisfying for almost all $t \geqslant t_*$ the requirement of inclusion

$$dx[t] \,/\, dt \in F(t, x[t]),\ x[t_*] = x_*,\ t \geqslant t_*$$

here $F(t,\ x)$ being the convex hull of the set of all vectors of the form $f = f(t, x, u, v)$, where $u \in P$, $v \in Q$. The motion $x[t;\ t_*,\ x_*,\ U,\ V]$ is defined in a similar manner to that used for obtaining $x[t;\ t_*,\ x_*,\ U,\ V_\tau]$ viz., by taking $x_\Delta[t]$ to the limit.

Only now the relation (1.3) for $x_\Delta[t]$ can be replaced by

$$\frac{dx_\Delta[t]}{dt} = \iint f(t, x_\Delta[t], u, v)\, \mu(du; p[\tau_i])\, \nu(dv; p[\tau_i])$$

$$x_\Delta[t_*] = x_*, \quad \tau_i \leqslant t < \tau_{i+1}, \quad i = 0, 1, 2, \ldots \tag{1.6}$$

where

$$\mu(du;\ p[\tau_i]) \in U(\tau_i,\ x_\Delta[\tau_i]),\ \ \nu(dv;\ p[\tau_i]) \in V(\tau_i,\ x_\Delta[\tau_i])$$

This completes the formal representation of the strategies of the players and the corresponding motions of the system (1.1).

A brief explanation of the concepts introduced is as follows. Let us e.g. find the motion $x[t;\ t_*\ x_*,\ U,\ V]$, which was introduced above as the limit of a sequence of approximate motions $x_{\Delta_k}[t] = x_{\Delta_k}[t;\ t_*,\ x_k,\ U,\ V]$, corresponding to piecewise continuous measures

$\mu(du;\ p[\tau_i]) \in U(\tau_i,\ x_{\Delta_k}[\tau_i])$ and $\nu(dv;\ p[\tau_i]) \in V(\tau_i, x_{\Delta_k}[\tau_i])$ for $\tau_i \leqslant t < \tau_{i+1}$.

We note that the mixing of the controls u and v, defined over the interval $[\tau_i,\ \tau_{i+1})$ by the measures $\mu(du;\ p[\tau_i])$ and $\nu(dv;\ p[\tau_i])$, respectively, can be performed approximately by mixing the controls u and v defined by

$$u[t] = u^{(j)} \text{ when } t \in [\tau_i^{(j)}, \tau_i^{(j+1)}), \; j = 1, \ldots, m^{(i)}$$

$$v[t] = v^{(s)} \text{ when } \in [\tau_i^{(s)}, \tau_i^{(s+1)}), \; s = 1, \ldots, l^{(i)}$$

over time. The values of the vectors $u^{(j)}$ and $v^{(s)}$ and the system of nonoverlapping semiopen subintervals $[\tau_i^{(j)}, \tau_i^{(j+1)})$ and $[\tau_i^{(s)}, \tau_i^{(s+1)})$ covering the interval $[\tau_i, \tau_{i+1})$ are defined by corresponding measures $\mu\ (du;\ p\ [\tau_i])$ and $\nu\ (dv;\ p\ [\tau_i])$, respectively.

In this way the motion $x\ [t;\ t_*, x_*,\ U,\ V]$ can be determined as a limit of certain motions of the system (1.1) accomplished in a defined manner by the controls of the first and second player mixed over time. It must be assumed, that the second (first) player is not aware of the actual method of selection of the semiopen intervals $[\tau_i^{(j)} \tau_i^{(j+1)})$ and $[\tau_i^{(s)}, \tau_i^{(s+1)})$ although he may have a knowledge of the strategy U (strategy V) of mixing (relation between the measures of the semiopen intervals $[\tau_i^{(j)}, \tau_i^{(j+1)})$ and $([\tau_i^{(s)}, \tau_i^{(s+1)}))$ and of the values of $u^{(j)}\ (v^{(s)})$ corresponding to the measure

$$\mu\ (du;\ p[\tau_i]) \in U\ (\tau_i,\ x_{\Delta k}[\tau_i])\ (\nu\ (dv;\ p\ [\tau_i]) \in V\ \ (\tau_i,\ x_{\Delta_k}[\tau_i]).$$

It follows, therefore, that the mixing of the controls u and v must be independent (in the sense assumed in the well-known game situations interpreted on the basis of the theory of probability). These assumptions correspond to the character of information described above and which are available to the players, i. e. the player is ignorant of the realization $u\ [t]$ or $v\ [t]$ of the controls chosen by his partner at a given instant (and later), and only knows the position $p\ [t] = \{t,\ x\ [t]\}$ realized.

In addition the following may be noted. The set of motions $x\ [t;\ t_*,\ x_*,\ U,\ \ V_\tau]$ contains any motion of the system (1.1) satisfying the condition $x\ [t_*] = x_*$, which may be realized with the strategy $U = U\ (t,\ x)$ chosen by the first player together with any strategy of the second player. Therefore, when it is stated that a certain condition holds for all motions $x\ [t;\ t_*,\ x_*,\ U,\ V_\tau]$ it means that the strategy $U = U\ (t\ ,\ x)$ guarantees that this condition is satisfied irrespective of any permitted behavior of the partner. Similar statement can be made concerning the motions $x\ [t;\ t_*,\ x_*,\ U_\tau,\ V]$.

Let us now introduce some notation and formulate the alternative mentioned previously. Let $x\ [t]$ by some motion of the system (1.1) and G a closed set belonging to the vector space $p = \{t,\ x\}$. Denote by $\vartheta\ (x\ [\cdot];\ G)$ the instant at which the point $p\ [t] = \{t,\ x\ [t]\}$ first reaches the set G and assume that $\vartheta\ (x\ [\cdot];\ G) = \infty$ if the condition $p\ [t] \in G$ does not hold for any $t \geqslant t_0$.

In the following $\rho\ (\{t,\ x\},\ G)$ denotes the Euclidean distance between the point $p = \{t,\ x\}$ and the set G. The instant at which the inequality $\rho\ (\{t,\ x\ [t]\}, G) \geqslant \varepsilon$ becomes valid for some motion $x\ [t]$ for the first time is denoted by $\tau^\varepsilon\ (x\ [\cdot];\ G)$, and the closed neighborhood ε of the set G is denoted by G^ε. Therefore $G^\varepsilon = \{p = = g + q : g \in G,\ \|q\| \leqslant \varepsilon\}$.

Here and in the followng $\|q\|$ denotes the Euclidean norm of the vector q.

The following statement expresses the basic result of this paper.

The Alternative. Let $p_0 = \{t_0,\ x_0\}$ be the initial position of the game, M and D some closed sets in the vector space $p = \{t,\ x\}$ and $\vartheta \geqslant t_0$ a finite number. Then one of the following two statements is true :— either

a) there exists a mixed strategy $U = U\ (t,\ x)$ of the first player such that the relations

$$\vartheta(x[\cdot];\ M) \leqslant \vartheta, \qquad \{t,\ x[t]\} \in D \text{ when } t_0 \leqslant t \leqslant \vartheta(x[\cdot];\ M)$$

hold for any motion

$$x[t] = x[t;\ \ t_0,\ x_0,\ U,\ V_\tau]$$

i.e. for the motion $x[t] = x[t;\ t_0,\ x_0,\ U,\ V_\tau]$ the condition $\{t,\ x[t]\} \in M$ will become valid not later than at the instant ϑ, and the phase restriction $\{t,\ x[t]\} \in D$ applies throughout the motion of the point $p[t] = \{t,\ x[t]\}$ from $p_0 = \{t_0,\ x_0\}$ to the set M, or

b) there exists a mixed strategy $V = V(t,\ x)$ of the second player and a positive number $\varepsilon > 0$ such that the condition

$$\rho(\{t,\ x[t]\},\ M) > \varepsilon \quad \text{when } t_0 \leqslant t \leqslant \min\ \ \{\vartheta,\ \tau^\varepsilon(x[\cdot];\ D)\}$$

holds for any motion $x[t] = x[t;\ t_0,\ x_0,\ U_\tau,\ V]$, i.e. no motions $x[t] = x[t;\ t_0,$ $x_0,\ U_\tau,\ V]$ exist satisfying the condition $\{t,\ x[t]\} \in D^\varepsilon$ and reaching M^ε not later than at the instant ϑ.

2. Since the approach to the problems investigated in this paper is based on the extremal construction used in [5–7], the basic elements of this construction should be defined.

Definition 2.1. Let there be a one-to-one correspondence between all values of t belonging to some interval $[t_0,\ \eta]$ in the phase space $\{x\}$ and nonempty sets $W(t)$. The collection of sets $W(t)\ (t_0 \leqslant t \leqslant \eta)$ shall be called u-stable in M if for any $t_* \in [t_0,\ \eta),\ x_* \in W(t_*)$ and $\delta \in (0,\ \eta - t_*]$ a motion $x[t] = x[t;\ t_*,\ x_*,$ $U_\tau,\ \nu(dv)]$ can be found for any value of the measure $\nu(dv)$, satisfying either the condition that $x[t_* + \delta] \in W(t_* + \delta)$, or the condition that $\{\tau,\ x[\tau]\} \in M$ for some $\tau \in [t_*,\ t_* + \delta]$.

Definition 2.2. Let a closed set G be defined in the vector space $p = \{t,\ x\}$, and a system of nonempty closed sets $W(t)\ (t_0 \leqslant t \leqslant \eta)$ be given. This collection will be called v-stable in G, if for any $t_* \in [t_0,\ \eta),\ x_* \in W(t_*)$ and $\delta \in (0,$ $\eta - t_*]$ a motion $x[t] = x[t;\ t_*,\ x_*,\ \mu(du),\ V_\tau]$ can be found for any value of the measure $\mu(du)$ satisfying either the inclusion $x[t_* + \delta] \in W(t_* + \delta)$ or the condition $\{\tau,\ x[\tau]\} \in G$ for some $\tau \in [t_*,\ t_* + \delta]$.

Let a collection of closed sets $W(t)\ (t_0 \leqslant t \leqslant \vartheta)$ be given in the space $\{x\}$, and assume that these sets can, in general, be empty for some $t \in [t_0,\ \vartheta]$. Now introduce the notion of a mixed strategy $U^{(e)} = U^{(e)}(t,\ x)$ of the first player extremal to this collection of sets. Denote by $\psi^*(t,\ x,\ s)$ the quantity given by

$$\psi^*(t,\ x,\ s) = \min_{\nu(dv)} \max_{\mu(du)} \iint s' f(t,\ x,\ u,\ v)\, \mu(du)\, \nu(dv) =$$
$$= \max_{\mu(du)} \min_{\nu(dv)} \iint s' f(t,\ x,\ u,\ v)\, \mu(du)\, \nu(dv) \qquad (2.1)$$

where s is an n-dimensional vector and where the prime denotes transposition, and calculate the maximum and the minimum over all possible regular Borel measures $\mu(du)$ and $\nu(dv)$ normed on P and Q, respectively. The validity of (2.1) has been proved previously e.g. in [11], p. 95. Let us assume first that at the point $t \in [t_0,\ \vartheta]$ under consideration the set $W(t)$ is nonempty and let $S(t,\ x)$ denote the set of all vectors of the form $s^\circ = w^\circ(x) - x$ where $w^\circ(x)$ are the points of the set $W(t)$ nearest to x. (If $x \in W(t)$, the set $S(t,\ x)$ obviously consists of a single null vector). When the set $W(t)$ is nonempty, $U^{(e)}(t, x)$ is defined as aggregate of all measures $\mu^\circ(du)$ satisfying the condition

$$\min_{\nu(dv)} \iint s^{\circ\prime} f(t, x, u, v)\, \mu^\circ(du)\, \nu(dv) = \psi^*(t, x, s^\circ) \tag{2.2}$$

for at least one vector s° from $S(t, x)$.

The existence of such measures follows from the relation (2.1). If however the set $W(t)$ is empty for some $t \in [t_0, \vartheta]$, then it must be assumed that $U^{(e)}(t, x)$ is composed, for any value of x, of all possible regular Borel measures $\mu(du)$ normed on P.

The function $U^{(e)} = U^{(e)}(t, x)$ is thus defined for all x and for $t \in [t_0, \vartheta]$; it also satisfies the condition of weak semicontinuity in x.

Let the mixed strategy of the first player, as given by the function $U^{(e)} = U^{(e)}(t, x)$, be extremal to the system of sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$.

The mixed strategy of the second player $V^{(e)} = V^{(e)}(t, x)$, extremal to some system of closed sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$, is defined in a similar manner. When $W(t)$ is nonempty, the set $V^{(e)}(t, x)$ is composed of the measures $\nu^\circ(dv)$ satisfying the con- the condition

$$\min_{\mu(du)} \iint s^{\circ\prime} f(t, x, u, v)\, \mu(du)\, \nu^\circ(dv) = \psi_*(t, x, s^\circ) \tag{2.3}$$

which must hold for at least one vector s° from $S(t, x)$. The quantity $\psi_*(t, x, s)$ is given here by

$$\psi_*(t, x, s) = \max_{\nu(dv)} \min_{\mu(du)} \iint s' f(t, x, u, v)\, \mu(du)\, \nu(dv) =$$

$$= \min_{\mu(du)} \max_{\nu(dv)} \iint s' f(t, x, u, v)\, \mu(du)\, \nu(dv) \tag{2.4}$$

where as before $\mu(du)$ and $\nu(dv)$ denote all possible regular Borel measures normed on P and Q, respectively.

The following two statements are valid.

L e m m a 2.1. If $x_0 \in W(t_0)$ and the collection of nonempty closed sets $W(t)$ $(t_0 \leqslant t \leqslant \eta)$ is u-stable in M, then the mixed strategy of the first player $U^{(e)} = U^{(e)}(t, x)$ extremal to the system of sets $W(t)$ $(t_0 \leqslant t \leqslant \eta)$ ensures that the condition

$$x[t] \in W(t) \quad \text{when } t_0 \leqslant t \leqslant \min\{\eta, \vartheta(x[\cdot]; M)\} \tag{2.5}$$

holds for any motion $x[t] = x[t; t_0, x_0, U^{(e)}, V_\tau]$.

L e m m a 2.2. If the collection of nonempty closed sets $W(t)$ $(t_0 \leqslant t \leqslant \eta)$ is v-stable in G and $x_0 \in W(t_0)$, then the mixed strategy of the second player $V^{(e)} = V^{(e)}(t, x)$ extremal to the system of sets $W(t)$ $(t_0 \leqslant t \leqslant \eta)$ ensures that the condition

$$x[t] \in W(t) \quad \text{when } t_0 \leqslant t \leqslant \min\{\eta, \vartheta(x[\cdot]; G)\}$$

holds for any motion $x[t] = x[t; t_0, x_0, U_\tau, V^{(e)}]$.

P r o o f o f L e m m a 2.1. Let $x[t] = x[t; t_*, x_*, U^{(e)}, V_\tau]$ be any motion corresponding to the strategies $U^{(e)}$ and V_τ. It will be shown that condition (2.5) holds for this motion. Let $x_{\Delta_k}[t]$ $(t_0 \leqslant t \leqslant \eta)$ denote a sequence of approximate motions whose limit is the generalized motion under consideration. i. e.

$$x_{\Delta_k}[t] = x_{\Delta_k}[t; t_0, x_k, U^{(e)}, V_\tau] \quad (t_0 \leqslant t \leqslant \eta)$$

Moreover,

$$\lim_{k\to\infty} \sigma(\Delta_k) = 0, \quad \lim_{k\to\infty} x_k = x_0, \quad \lim_{k\to\infty} \left(\max_{t_0 \leqslant t \leqslant \eta} \|x[t] - x_{\Delta_k}[t]\| \right) = 0 \tag{2.6}$$

It can be easily shown that the requirement (2.5) holds for the motion $x[t]$ under consideration is equivalent to the statement that for any arbitrarily small $\alpha > 0$ a number K can be found such that the relation

$$x_{\Delta_k}[t] \in W^{\alpha}(t) \text{ when } t_0 \leqslant t \leqslant \min\{\eta, \vartheta(x_{\Delta_k}[\cdot]; M^{\alpha})\} \tag{2.7}$$

holds for all $k \geqslant K$ for the motions $x_{\Delta_k}[t]$. Consequently the proof of Lemma 2.1 is reduced to a proof of the above assertion. Let us introduce the quantity $\varepsilon_k[t] = \rho(x_{\Delta_k}[t], W(t))$, remembering that $\rho(x, W)$ is the distance between the point x and the set W. It will be shown that the quantity $\varepsilon_k[t]$ at sufficiently large k doesn't exceed any predetermined number $\alpha > 0$ for all $t \in [t_0, \min\{\eta, \vartheta(x_{\Delta_k}[\cdot]; M^{\alpha})\}]$, i.e. condition (2.7) holds.

Thus, let us see how the quantity $\varepsilon_k[t]$ varies when $\tau_i^{(k)} \leqslant t < \tau_{i+1}^{(k)}$, $\tau_i^{(k)} < \eta$ where $[\tau_i^{(k)}, \tau_{i+1}^{(k)})$ are semi-intervals of the range set Δ_k. The motion $x_{\Delta_k}[t]$ for $t \in [\tau_i^{(k)}, \tau_{i+1}^{(k)})$ is given by

$$\frac{dx_{\Delta_k}[t]}{dt} = f^{(e)}[t], \quad f^{(e)}[t] \in F(t, x_{\Delta_k}[t]; \mu^{\circ}(du; p[\tau_i^{(k)}]))$$

$$\mu^{\circ}(du; p[\tau_i^{(k)}]) \in U^{(e)}(\tau_i^{(k)}, x_{\Delta_k}[\tau_i^{(k)}])$$

i.e. the measure $\mu^{\circ}(du; p[\tau_i^{(k)}])$ satisfies the condition (2.2) in which the vector s° is given by

$$s^{\circ} = s^{\circ}[\tau_i^{(k)}] = w^{\circ} - x_{\Delta_k}[\tau_i^{(k)}] \tag{2.8}$$

where w° is a certain point of the aggregate of points nearest to $x_{\Delta_k}[\tau_i^{(k)}]$ and belonging to the set $W(\tau_i^{(k)})$. Assuming now that the inequality

$$\varepsilon_k[\tau_i^{(k)}] = \| w^{\circ} - x_{\Delta_k}[\tau_i^{(k)}] \| = \| s^{\circ}[\tau_i^{(k)}] \| > 0 \tag{2.9}$$

holds, let $\nu_0(dv)$ be the measure on Q conveying the minimum value to (2.2) for $t = \tau_i^{(k)}$, $x = x_{\Delta_k}[\tau_i^{(k)}]$ and $s^{\circ} = s^{\circ}[\tau_i^{(k)}]$, i.e.

$$\iint s^{\circ\prime}[\tau_i^{(k)}] f(\tau_i^{(k)}, x_{\Delta_k}[\tau_i^{(k)}], u, v)\, \mu^{\circ}(du; p[\tau_i^{(k)}])\, \nu_0(dv) = \psi^*(\tau_i^{(k)}, x_{\Delta_k}[\tau_i^{(k)}], s^{\circ}[\tau_i^{(k)}]) \geqslant$$
$$\geqslant \iint s^{\circ\prime}[\tau_i^{(k)}] f(\tau_i^{(k)}, x_{\Delta_k}[\tau_i^{(k)}], u, v)\, \mu(du)\, \nu_0(dv) \tag{2.10}$$

Using the condition of u-stability of the system of sets $W(t)$ $(t_0 \leqslant t \leqslant \eta)$, and assuming first that the point $p[t] = \{t, x[t]\}$ does not reach the set M when $t \in [\tau_i^{(k)}, \tau_{i+1}^{(k)}]$ for any motion $x[t] = x[t; \tau_i^{(k)}, w^{\circ}, U_{\tau}, \nu_0(dv)]$, we find, that a motion $x_*[t] = x[t; \tau_i^{(k)}, w^{\circ}, U_{\tau}, \nu_0(dv)]$ exists such that

$$x_*[\tau_{i+1}^{(k)}] \in W(\tau_{i+1}^{(k)}) \tag{2.11}$$

This motion is given by

$$\frac{dx_*[t]}{dt} = f_*[t], \quad x_*[\tau_i^{(k)}] = w^{\circ}, \quad f_*[t] = F(t, x_*[t]; \nu_0(dv))$$

The inequality

$$\varepsilon_k[\tau_{i+1}^{(k)}] = \rho(x_{\Delta_k}[\tau_{i+1}^{(k)}], W(\tau_{i+1}^{(k)})) \leqslant \| x_{\Delta_k}[\tau_{i+1}^{(k)}] - x_*[\tau_{i+1}^{(k)}] \| \tag{2.12}$$

follows from (2.11).

Let us secondly estimate the distance between the points $x_{\Delta_k}[\tau_{i+1}^{(k)}]$ and $x_*[\tau_{i+1}^{(k)}]$. Here the following relations shall be used which follow from the fact that the Lipshits condition holds with respect to x for the right-hand side of (1.1)

$$\begin{matrix} F(t, x + \Delta x; \mu(du)) \in F^{\beta}(t, x; \mu(du)) \\ F(t, x + \Delta x; \nu(dv)) \in F^{\beta}(t, x, \nu(dv)) \end{matrix} \begin{pmatrix} \beta = \lambda \| \Delta x \| \\ (\lambda = \text{const} > 0) \end{pmatrix} \tag{2.13}$$

where F^{β} denotes the β-neighborhood of the set F. The relations (2.13) hold for any measures $\mu\,(du)$ and $\nu\,(dv)$ for all t and x.

Using (2.13), and taking into account the fact that the sets $F\,(t,\ x;\ \mu\,(du))$ and $F\,(t,\ x;\ \nu\,(dv))$ are convex and depend continuously on t, we obtain the following expressions:

$$x_{\Delta_k}[\tau_{i+1}^{(k)}] = x_{\Delta_k}[\tau_i^{(k)}] + f^{(e)}\delta_i^{(k)} + o\,(\delta_i^{(k)}) \tag{2.14}$$

$$x_*[\tau_{i+1}^{(k)}] = w^\circ + f_*\delta_i^{(k)} + o\,(\delta_i^{(k)})$$

where

$$\delta_i^{(k)} = \tau_{i+1}^{(k)} - \tau_i^{(k)},\ f^{(e)} \in F\,(\tau_i^{(k)},\ x_{\Delta_k}[\tau_i^{(k)}];\ \mu^\circ\,(du;\ p\,[\tau_i^{(k)}]))$$

$$f_* \in F\,(\tau_i^{(k)},\ w^\circ;\ \nu_0\,(dv)) \tag{2.15}$$

The symbol $o\,(\delta)$ denotes a higher order infinitesimal in δ. By (2.8), (2.9) and (2.12) relations (2.14) yield the following estimate:

$$\varepsilon_k^2[\tau_{i+1}^{(k)}] \leqslant \| x_{\Delta_k}[\tau_{i+1}^{(k)}] - x_*[\tau_{i+1}^{(k)}]\|^2 = \| s^\circ\,[\tau_i^{(k)}]\|^2 + 2\delta_i^{(k)} s^{\circ\prime}[\tau_i^{(k)}]\,(f_* - f^{(e)}) + o\,(\delta_i^{(k)}) \tag{2.16}$$

Finally the scalar product $s^{\circ\prime}\,[\tau_i^{(k)}]\ (f_* - f^{(e)})$ will be estimated. From (2.13) and (2.15) it follows that a vector $f^* \in F\,(\tau_i^{(k)},\ x_{\Delta_k}\,[\tau_i^{(k)}];\ \nu_0\,(dv))$ exists, satisfying the inequality $\|f_* - f^*\| \leqslant \lambda \| w^\circ - x_{\Delta_k}\,[\tau_i^{(k)}]\,\| = \lambda \| s^\circ\,[\tau_i^{(k)}]\|$. Therefore the relation

$$s^{\circ\prime}\,[\tau_i^{(k)}]\,f_* \leqslant s^{\circ\prime}\,[\tau_i^{(k)}]\,f^* + \lambda \|\,s^\circ\ [\tau_i^{(k)}]\,\|^2 \tag{2.17}$$

is valid.

At this stage it should be noted that for any vector $f \in F\,(t,\ x;\ \mu\,(du))$ $(f \in F\,(t,\ x;\ \nu\,(dv)))$ a measure $\nu\,(dv)$ $(\mu\,(du))$ can be chosen such that the equation

$$f = \iint f\,(t,\ x,\ u,\ v)\ \mu\,(du)\ \nu\,(dv)$$

holds.

When this is taken into account, the scalar products $s^{\circ\prime}\ [\tau_i^{(k)}\,]\ f^{(e)}$ and $s^{\circ\prime}\ [\tau_i^{(k)}]\ f^*$ can be written in the form

$$s^{\circ\prime}\,[\tau_i^{(k)}]\,f^{(e)} = \iint s^{\circ\prime}\,[\tau_i^{(k)}]\,f\ (\tau_i^{(k)},\ x_{\Delta_k}\,[\tau_i^{(k)}],\ u,\ v)\,\mu^\circ\,(du;\ p\,[\tau_i^{(k)}])\ \nu_*\,(dv)$$

$$s^{\circ\prime}\,[\tau_i^{(k)}]\,f^* = \iint s^{\circ\prime}\,[\tau_i^{(k)}]\,f\,(\tau_i^{(k)},\ x_{\Delta_k}\,[\tau_i^{(k)}],\ u,\ v)\ \mu_*\,(du)\ \nu_0\,(dv),$$

where $\mu_*\ (du)$ and $\nu_*\ (dv)$ are some measures on P and Q, respectively. Since the measure $\mu^\circ\ (du;\ p\ [\tau_i^{(k)}])$ belongs to the set $U^{(e)}\ (\tau_i^{(k)},\ x_{\Delta_k}\ [\tau_i^{(k)}])$, i.e. it satisfies the condition (2.2), while the measure $\nu_0\ (dv)$ is given by (2.10), the following inequalities hold:

$$s^{\circ\prime}\,[\tau_i^{(k)}]\,f^{(e)} \geqslant \psi^*\,(\tau_i^{(k)},\ x_{\Delta_k}\,[\tau_i^{(k)}],\ s^\circ\ [\tau_i^{(k)}])$$

$$s^{\circ\prime}\,[\tau_i^{(k)}]\,f^* \leqslant \psi^*\,(\tau_i^{(k)},\ x_{\Delta_k}\,[\tau_i^{(k)}],\ s^\circ\ [\tau_i^{(k)}])$$

From (2.17) the latter yields the estimate

$$s^{\circ\prime}\,[\tau_i^{(k)}]\,(f_* - f^{(e)}) \leqslant \lambda \|\,s^\circ\ [\tau_i^{(k)}]\,\|^2 = \lambda \varepsilon_k^2\ [\tau_i^{(k)}] \tag{2.18}$$

Substituting (2.18) into (2.16) we obtain the inequality

$$\varepsilon_k^2\,[\tau_{i+1}^{(k)}] \leqslant \varepsilon_k^2\,[\tau_i^{(k)}]\,(1 + 2\delta_i^{(k)}\lambda) +\ o\,(\delta_i^{(k)})$$

The above discussion clearly shows that the ratio $o\ (\delta_i^{(k)})\ /\ \delta_i^{(k)}$ which depends, in general, on the position $p\ [\tau_i^{(k)}] = \{\tau_i^{(k)},\ x_{\Delta_k}\ [\tau_i^{(k)}]\}$ of the game, tends to zero as $\delta_i^{(k)} \to 0$ uniformly in $p \in \Gamma$, where Γ is an arbitrary bounded region.

Furthermore, this statement remains valid even when $\tau_{i+1}^{(k)}$ is replaced by any t from the interval $[\tau_i^{(k)},\ \tau_{i+1}^{(k)}]$. Thus we have

$$\varepsilon_k^2\ [t] \leqslant \varepsilon_k^2\,[\tau_i^{(k)}]\,(1 + 2\delta_i^{(k)}\lambda) + \ o\ (\delta_i^{(k)})\ \text{ when }\ t \in [\tau_i^{(k)},\ \tau_{i+1}^{(k)}] \tag{2.19}$$

We note that this inequality is obtained on the basis of the assumption that the condition $\{t, x[t]\} \notin M$ when $t \in [\tau_i^{(k)}, \tau_{i+1}^{(k)}]$ is satisfied for any motion $x[t] = x[t; \tau_i^{(k)}, w^\circ, U_\tau, \nu_0(dv)]$.

Two cases are possible: either the latter assumption holds for the approximate motion $x_{\Delta_k}[t]$ over any interval $[\tau_i^{(k)}, \tau_{i+1}^{(k)})$, $\tau_i^{(k)} < \eta$, or there exist some interval $[\tau_j^{(k)}, \tau_{j+1}^{(k)})$, $\tau_j^{(k)} < \eta$ when this assumption breaks down for the first time. In this latter case, however, it may be easily shown that the following relation holds for some $t_* \in [\tau_j^{(k)}, \tau_{j+1}^{(k)})$ $(t_* \leqslant \eta)$

$$\rho(\{t_*, x_{\Delta_k}[t_*]\}, M) \leqslant \varepsilon_k[\tau_j^{(k)}] + \omega(\delta_j^{(k)}) \tag{2.20}$$

$$\omega(\delta) \to 0 \quad \text{as} \quad \delta \to 0$$

and the estimate (2.19) remains valid for all $i = 0, 1, \ldots, j-1$.

Let an arbitrary number $\alpha > 0$ now be chosen. Using the estimate (2.19) and taking into account the fact that $\lim_{k\to\infty} \varepsilon_k[t_0] = 0$ which by virtue of (2.6) follows from the condition $x_0 \in W(t_0)$, it can be proved that a number $K > 0$ exists such that for all $k \geqslant K$ the inequality

$$\varepsilon_k[t] \leqslant {}^1/_2\alpha \qquad \text{as} \quad t_0 \leqslant t \leqslant \min\{\eta, \tau_j^{(k)}\} \tag{2.21}$$

$$\omega(\delta_j^{(k)}) \leqslant \omega(\sigma(\Delta_k)) \leqslant {}^1/_2\alpha$$

holds.

Consequently, from the definition of $\varepsilon_k[t]$ and by virtue of (2.20), condition (2.7) holds. This proves the validity of the statement made at the beginning of this proof and thus completes the proof of Lemma 2.1.

Lemma 2.2 is proved in a similar fashion.

3. Let us now obtain a system of sets $W(t)$ maximal in some particular sense, possessing the property of u-stability in M. The study of such a system will establish the validity of the alternative expressed in Sect. 1.

Let $\varkappa(x[\cdot]; \tau, \vartheta)$ denote a functional defined on continuous vector functions $x[t]$ $(\tau \leqslant t \leqslant \vartheta)$ given by

$$\varkappa(x[\cdot]; \tau, \vartheta) = \min_t \rho(\{t, x[t]\}, M) + \max_t \rho(\{t, x\}, D) \tag{3.1}$$

where the minimum and the maximum are computed over $\tau \leqslant t \leqslant \vartheta$ and $\tau \leqslant t \leqslant \min\{\vartheta, \vartheta(x[\cdot]; M)\}$, respectively.

We note that the functional $\varkappa(x[\cdot]; \tau, \vartheta]$ is lower semicontinuous, i.e. for any continuous vector function $x_*[t]$ $(\tau \leqslant t \leqslant \vartheta)$ and for any $\varepsilon > 0$, values of $\delta > 0$ can be found such that the inequality

$$\varkappa(x[\cdot]; \tau, \vartheta) \geqslant \varkappa(x_*[\cdot]; \tau, \vartheta) - \varepsilon$$

holds for any vector function $x[t]$ $(\tau \leqslant t \leqslant \vartheta)$ satisfying the condition

$$\max_t \|x[t] - x_*[t]\| \leqslant \delta \qquad (\tau \leqslant t \leqslant \vartheta)$$

Definition 3.1. We say that the set M is positionally absorbed by D from the position $p_* = \{t_*, w_*\}$ and the instant ϑ if for any mixed strategy $V = V(t, x)$ of the second player, a motion $x[t] = x[t; t_*, w_*, U_\tau, V]$ can be found such that

$$\varkappa(x[\cdot]; t_*, \vartheta) = 0 \tag{3.2}$$

i.e. $\vartheta(x[\cdot]; M) \leqslant \vartheta$ and $\{t, x[t]\} \in D$ for all $t \in [t_*, \vartheta(x[\cdot]; M)]$.

Let $W(t, \vartheta)$ denote the set of all points w such that the set M is positionally absorbed by D from the position $p = \{t, w\}$ at the instant ϑ.

Let us consider some properties of the sets $W(t, \vartheta)$. First the following auxilliary statement will be proved.

Lemma 3.1. Let $p_* = \{t_*, w_*\}$ be some position of the game, where $t_* \leqslant \vartheta$ and $\{t_*, w_*\} \in D$. If both a measure $\nu_*(dv)$ on Q and an instant $t^* \in [t_*, \vartheta]$ exist such that the conditions

$$\rho(\{t, x[t]\}; M) > 0 \quad \text{when } t \in [t_*, t^*] \tag{3.3}$$

$$x[t^*] \notin W(t^*, \vartheta) \tag{3.4}$$

hold for any motion $x[t] = x[t; t_*, w_*, U_\tau, \nu_*(dv)]$, then the set $W(t_*, \vartheta)$ does not contain the point w_*.

Proof. From the definition of the set $W(t, \vartheta)$ it may be inferred that the validity of Lemma 3.1 will be established if it can be proved that a mixed strategy $V_* = V_*(t, x)$ of the second player exists such that the inequality

$$\varkappa(x[\cdot]; t_*, \vartheta) > 0 \tag{3.5}$$

holds for any motion $x[t] = x[t; t_*, w_*, U_\tau, V_*]$.

The existence of this mixed strategy V_* is proved as follows. First, we assume that some system of sets $W_*(t)$ $(t_* \leqslant t \leqslant \vartheta)$ possessing the properties set forth below exists. Then we shall show that in this case a mixed strategy of the second player extremal to such a system secures the validity of the inequality (3.5). Finally, we shall show that when the conditions of Lemma 3.1 are satisfied, a system of sets $W_*(t)$ $(t_* \leqslant t \leqslant \vartheta)$ possessing the required properties exists.

Thus, we assume that there exists a system of sets $W_*(t)$ possessing the following properties.

1°. The sets $W_*(t)$ are nonempty and closed for $t_* \leqslant t \leqslant \eta$, where η is a number which satisfies the inequality $\eta \leqslant \vartheta$. If $\eta < \vartheta$, the equality

$$\rho(\{\eta, w\}; D) = \varepsilon > 0 \tag{3.6}$$

holds for any point $w \in W_*(\eta)$.

2°. The inequality

$$\rho(\{t, w\}, M) \geqslant \varepsilon > 0 \tag{3.7}$$

holds for any point $w \in W_*(t)(t_* \leqslant t \leqslant \eta)$.

3°. The point w_* belongs to the set $W_*(t_*)$.

4°. The system of sets $W_*(t)$ $(t_* \leqslant t \leqslant \eta)$ is v-stable in G, where G is an aggregate of points satisfying the condition

$$\rho(\{t, x\}, D) \geqslant \varepsilon > 0$$

We shall show that the inequality $\varkappa(x[\cdot]; t_*, \vartheta) \geqslant \varepsilon > 0$ holds for any motion $x[t] = x[t; t_*, w_*, U_\tau, V^{(e)}]$. (Here $V^{(e)} = V^{(e)}(t, x)$ is the mixed strategy of the second player extremal to the system of sets $W_*(t)$ $(t_* \leqslant t \leqslant \vartheta)$.

Indeed, by virtue of Lemma 2.2, the condition

$$x[t] \in W_*(t), \qquad t_* \leqslant t \leqslant \min\{\eta, \vartheta(x[\cdot]; G)\} \tag{3.8}$$

holds for any motion $x[t] = x[t; t_*, w_*, U_\tau, V^{(e)}]$.

Let us assume that $\min\{\eta, \vartheta(x[\cdot]; G)\} = \vartheta$, then (3.8) and (3.7) yield the inequality

$$\rho(\{t, x[t]\}, M) \geqslant \varepsilon > 0 \quad \text{for } t_* \leqslant t \leqslant \vartheta \tag{3.9}$$

If, on the other hand, $\xi = \min\{\eta, \vartheta(x[\cdot]; G)\} < \vartheta$, then by virtue of properties 1°, 2° and 4° from the condition (3.8) we have the following relations:

$$\rho(\{t,\ x[t]\},\ M) \geqslant \varepsilon > 0 \quad \text{for} \quad t \in [t_*,\ \xi]$$
$$\rho(\{\xi,\ x[\xi]\},\ D) = \varepsilon > 0 \tag{3.10}$$

Thus either (3.9) or (3.10) holds for any motion $x[t] = x[t;\ t_*,\ w_*,\ U_\tau,\ V^{(\varepsilon)}]$; therefore the inequality

$$\varkappa(x[\cdot];\ t_*,\ \vartheta) \geqslant \varepsilon > 0 \tag{3.11}$$

holds for any motion $x[t] = x[t,\ t_*,\ w_*,\ U_\tau,\ V^{(\varepsilon)}]$.

It may also be shown that under the conditions of Lemma 3.1 a system of sets $W_*(t)$ $(t_* \leqslant t \leqslant \eta)$ may be constructed with the properties $1^\circ - 4^\circ$ listed above.

Let X be the set of points $x[t^*] = x[t^*,\ t_*, w_*,\ U_\tau,\ \nu_*(dv)]$ corresponding to every possible motions $x[t] = x[t; t_*,\ w_*, U_\tau,\ \nu_*\ (dv)]$. The set X is bounded, closed, and by virtue of (3.4) the intersection of X and $W(t^*,\ \vartheta)$ is empty. This means that for any point $w \in X$, a mixed strategy $V_w = V_w(t, x)$ of the second player can be found such that the inequality $\varkappa(x[\cdot];\ t^*,\ \vartheta) > 0$ will hold for any motion $x[t] = x[t;\ t^*, w,\ U_\tau,\ V_w]$. The set of the vector functions $x[t;\ t^*,\ w,\ U_\tau,\ V_w]$ will be compact in itself, and the functional $\varkappa(x[\cdot];\ t^*, \vartheta)$ will be lower semicontinuous in $x[\cdot]$. A number $\alpha(w) > 0$ therefore exists such that the inequality

$$\varkappa(x[\cdot];\ t^*,\ \vartheta) \geqslant \alpha(w) > 0 \tag{3.12}$$

holds for all motions $x[t] = x[t;\ t^*,\ w,\ U_\tau,\ V_w]$.

Since the set of motions $x[t;\ t^*,\ w,\ U_\tau,\ V]$ depends on the initial condition $x[t^*] = w$ semicontinuously from above with respect to inclusion, from (3.12) it follows that the relation

$$\varkappa(x[\cdot];\ t^*,\ \vartheta) \geqslant {}^1\!/_2\alpha(w) > 0$$

holds for all points x^* satisfying the inequality $\|x^* - w\| \leqslant r(w)$ (where $r(w)$ is a positive number) and for any motion $x[t] = x[t;\ t^*,\ x^*,\ U_\tau,\ V_w]$.

A set of spheres

$$R(w,\ r(w)) = \{x : \|x - w\| < r(w),\ r(w) > 0\}$$

covers the closed bounded set X, and from this cover we can select a finite subset

$$R_i = R(w_i,\ r(w_i)) = \{x : \|x - w_i\| \leqslant r(w_i)\}.$$
$$X \subset \bigcup R_i \ (i = 1, \ldots, m) \tag{3.13}$$

In the following the strategies $V_w = V_w(t, x)$ corresponding to the points w_i $(i = 1, \ldots, m)$ will be denoted simply by $V_i = V_i(t, x)$. The aggregate of all motions $x[t] = x[t;\ t^*, x^*,\ U_\tau,\ V_i]$ $(t^* \leqslant t \leqslant \vartheta)$ corresponding to all possible initial conditions $x[t^*] = x^* \in R_i$ will be denoted by $X_i[t^*,\ \vartheta]$.

The symbol $X[t^*, \vartheta]$ will denote the set of all vector functions $x[t]$ $(t^* \leqslant t \leqslant \vartheta)$ defined by

$$X[t^*, \vartheta] = \bigcup X_i[t^*, \vartheta] \qquad (i=1, \ldots, m) \tag{3.14}$$

The set of motions $x[t;\ t^*, x^*,\ U_\tau,\ V_i]$ $(t^* \leqslant t \leqslant \vartheta,\ i = 1, \ldots, m)$ is compact in itself and depends on x^* semicontinuously from above with respect to inclusion, therefore the closure property of the sets R_i implies that the sets $X_i[t^*, \vartheta]$ and $X[t^*, \vartheta]$ are compact in themselves. From the structure of these sets it follows that the inequality

$$\varkappa(x[\cdot];\ t^*,\ \vartheta) \geqslant \alpha^* > 0 \tag{3.15}$$

holds for any vector function $x[t]$ $(t^* \leqslant t \leqslant \vartheta)$ belonging to the set $X[t^*, \vartheta]$. Here $\alpha^* = \min\{{}^1\!/_2\alpha\ (w_i)\}$ for $i = 1,\ \ldots,\ m$.

Let $X[t_*,\ \vartheta]$ denote a set of all continuous vector functions $x[t]$ $(t_* \leqslant t \leqslant \vartheta)$ which coincide with some of the motions $x[t;\ t_*,\ w_*,\ U_\tau,\ \nu_*(dv)]$ when $t_* \leqslant t \leqslant t^*$ and belong to the set $X[t^*, \vartheta]$ when $t^* \leqslant t \leqslant \vartheta$ From (3.3) and (3, 5) it follows that the inequality

$$\varkappa(x[\cdot];\ t_*,\ \vartheta) \geqslant \alpha_* > 0$$

holds for any vector function $x[t]$ $(t_* \leqslant t \leqslant \vartheta)$ belonging to the set $X[t_*, \vartheta]$.

Using the fact that the set $X[t_*, \vartheta]$ is compact in itself together with the latter inequality, it may be deduced that at least one of the following two relations hold for any vector function $x[t]$ $(t_* \leqslant t \leqslant \vartheta)$ belonging to the set $X[t_*, \vartheta]$:

$$\max_t \rho(\{t, x[t]\}, D) > \varepsilon \quad \text{for} \quad t_* \leqslant t \leqslant \min\{\vartheta,\ \vartheta(x[\cdot];\ M^\varepsilon)\} \tag{3.16}$$

$$\min_t \rho(\{t,\ x[t]\},\ M) > \varepsilon \quad \text{for} \quad t_* \leqslant t \leqslant \vartheta \tag{3.17}$$

where $\varepsilon > 0$ is a certain positive number.

Let $W_*(\tau)$ be a set of points belonging to the phase space $\{x\}$ defined as follows: a point w belongs to the set $W_*(\tau)$ if and only if a vector function $x[t]$ $(t_* \leqslant t \leqslant \vartheta)$ exists, belonging to the set $X[t_*, \vartheta]$ and satisfying the conditions

$$x[\tau] = w, \quad \max \rho(\{t,\ x[t]\},\ D) \leqslant \varepsilon \quad \text{for} \quad t_* \leqslant t \leqslant \tau \tag{3.18}$$

The system of sets $W_*(\tau)$ $(t_* \leqslant \tau \leqslant \vartheta)$ so constructed may be shown to possess the properties 1° – 4° listed above. Let denote by η the upper bound of the numbers $\tau \leqslant \vartheta$ for which the sets $W_*(\tau)$ are nonempty. In this case the set $W_*(\eta)$, as well as any of the sets $W_*(\tau)$ for $t_* \leqslant \tau \leqslant \eta$, may be shown to be nonempty.

Indeed, let τ_k $(k = 1, 2, \ldots)$ be a sequence of numbers converging from the left to η and let the set $W_*(\tau_k)$ $(k = 1, 2 \ldots)$ be nonempty.

Consider a certain sequence of points w_k

$$w_k \in W(\tau_k) \qquad (k = 1,\ 2,\ \ldots)$$

Each point w_k has a corresponding vector function $x_k[t]$ $(t_* \leqslant t \leqslant \vartheta)$ belonging to the set $X[t_*, \vartheta]$ and satisfying the condition

$$x_k[\tau_k] = w_k, \quad \max_t \rho(\{t,\ x_k[t]\},\ D) \leqslant \varepsilon \quad \text{for} \quad t_* \leqslant t \leqslant \tau_k \tag{3.19}$$

Out of the sequence vector functions $x_k[t]$ $(t_* \leqslant t \leqslant \vartheta)$ now choose a subsequence converging to some vector function $x_*[t]$ $(t_* \leqslant t \leqslant \vartheta)$ also belonging to the set $X[t_*, \vartheta]$. Taking into account the fact that $\tau_k \to \eta$ as $k \to \infty$, from (3.19) we obtain

$$x_*[\eta] = w_*, \quad \max_t \rho(\{t, x_*[t]\}, D) \leqslant \varepsilon \quad \text{for} \quad t_* \leqslant t \leqslant \eta$$

Thus a point $w^* = x_*[\eta]$ belonging to the set $W_*(\eta)$ exists, showing that $W_*(\eta)$ is therefore nonempty. It may easily be shown that any set $W_*(\tau)$ for $t_* \leqslant \tau \leqslant \eta$ will also be nonempty. Closure of the sets $W_*(\tau)$ $(t_* \leqslant \tau \leqslant \eta)$ can be verified using arguments similar to those employed above to show that $W_*(\eta)$ is nonempty. When $\eta < \vartheta$, the definition of η as the largest number of those τ for which $W(\tau)$ is nonempty yields the following relation: $\max_t \rho(\{t,\ x[t]\},\ D) = \rho(\{\eta,\ x[\eta]\},\ D) = \varepsilon, \quad t_* \leqslant t \leqslant \eta$

where $x[t]$ $(t_* \leqslant t \leqslant \vartheta)$ are the elements of $X[t_*, \vartheta]$ which, by (3.18), correspond to the points $w \in W_*(\eta)$. The last equation therefore implies the correctness of (3.6) which verifies the condition 1°.

Condition 2° follows directly from (3.16) – (3.18). The inclusion $w_* \in W_*(t_*)$ follows from the condition $\{t_*, x[t_*]\} \in D$ and from the relation $x[t_*] = w_*$ which obviously holds for all elements of the set $X[t_*, \vartheta]$.

To confirm that condition 4° holds for the system of sets $W_*(\tau)$ constructed we assume that w is any point belonging to the set $W_*(\tau)$ $(t_* \leqslant \tau < \eta)$, $\mu_*(du)$ is a certain measure on P, and δ is a number satisfying the condition $0 < \delta \leqslant \eta - \tau$. It can be shown that a motion $x^* [t] = x [t; \tau, w, \mu_* (du), V_\tau]$, $(\tau \leqslant t \leqslant \tau + \delta)$ exists for which one of the following two conditions holds:

$$x^* [\tau + \delta] \in W_* (\tau + \delta) \tag{3.20}$$
$$\max \rho (\{t, x^* [t]\}, D) \geqslant \varepsilon \quad \text{for} \quad \tau \leqslant t \leqslant \tau + \delta \tag{3.21}$$

Let $x_* [t]$ denote a vector function belonging to $X [t_*, \vartheta]$ and corresponding, by virtue of (3.18), to the point w, i.e. satisfying the conditions

$$\max_t \rho (\{t, x [t]\}, D) \leqslant \varepsilon \quad \text{for} \quad t_* \leqslant t \leqslant \tau$$
$$x_* [\tau] = w \tag{3.22}$$

It can be shown that amongst the motions $x [t; \tau, w, \mu_* (du), V_\tau]$ such a motion $x^* [t]$ exists that the continuous vector function

$$x [t] = \begin{cases} x_* [t] & \text{for} \quad t_* \leqslant t \leqslant \tau \\ x_* [t] & \text{for} \quad \tau \leqslant t \leqslant \vartheta \end{cases}$$

is an element of the set $X [t_*, \vartheta]$. Then, in this case for the motion denoted by $x^* [t] = x [t; \tau, w, \mu_* (du), V_\tau]$, one of the conditions (3.20) and (3.21) must hold.

Indeed, if $\max_t \rho (\{t, x^* [t]\}, D) \leqslant \varepsilon$ for $\tau \leqslant t \leqslant \tau + \delta$, then by (3.22) and the definition of the set $W_* (t)$ either the inclusion (3.20) or the inequality (3.21) holds.

The fourth property of the family of sets $W_* (t)$ $(t_* \leqslant t \leqslant \eta)$ is thus verified and this completes the proof of Lemma 3.1.

Lemma 3.2. Every set $W (t, \vartheta)$ $(t_0 \leqslant t \leqslant \vartheta)$ is closed. The sets $W (t, \vartheta)$ $(t_0 \leqslant t \leqslant \vartheta)$ satisfy the inclusions

$$W (t, \vartheta) \subset D^* (t), \quad D^* (t) \cap M^* (t) \subset W (t, \vartheta) \tag{3.23}$$

Here $D^* (t)$ and $M^* (t)$ are closed sets in the space $\{x\}$ and are defined by

$$D^* (t) = \{x\colon \{t, x\} \in D\}, \qquad M^* (t) = \{x : \{t, x\} \in M\}$$

The validity of inclusions (3.23) follow directly from the Definition 3.1 and hence it only remains to show that the set $W (t, \vartheta)$ is closed. Let us assume that the point w_* does not belong to the set $W (t_*, \vartheta)$ $(t_* \in [t_0, \vartheta])$. This means that a mixed strategy of the second player $V_* = V_* (t, x)$ exists for which the inequality $\varkappa (x [\cdot]; t_*, \vartheta) > 0$ holds for any motion $x [t] = x [t; t_*, w_*, U_\tau, V_*]$.

The set of vector functions $x [t; t_*, w_*, U_\tau, V_*]$ $(t_* \leqslant t \leqslant \vartheta)$, compact in itself, depends on the initial condition w_*, semicontinuously from above with respect to inclusion, and the functional $\varkappa (x [\cdot]; t_*, \vartheta)$ is lower semicontinuous; therefore $\varepsilon > 0$ exists such that the inequality $\varkappa (x [\cdot]; t_*, \vartheta) > 0$ holds for any motion $x [t; t_*, w, U_\tau, V_*]$ where $\| w - w_* \| \leqslant \varepsilon$. Consequently a ε-neighborhood $R (w_*, \varepsilon)$ $(\varepsilon > 0)$ can be found for any point $w_* \notin W (t_*, \vartheta)$ such that $R (w_*, \varepsilon) \cap W (t_*, \vartheta) = \phi$, i.e. the complement of the set $W (t_*, \vartheta)$ is open and hence the set $W (t_*, \vartheta)$ is closed thus proving Lemma 3.2.

Let us denote, as before, a closed sphere of radius r, with its center at the point x_0, by $R (x_0, r)$. The following statement holds.

Lemma 3.3. If $x_0 \in W (t_0, \vartheta)$, then for any $r > 0$ a function $\eta (r)$ $(t_0 \leqslant \eta (r) \leqslant \vartheta)$ may be found for which the following conditions hold:

1) every one of the sets $W (t, \vartheta) \cap R (x_0, r)$ is nonempty when $t_0 \leqslant t \leqslant \eta (r)$;

2) either $$\rho(x_0, W (\eta (r), \vartheta)) = r$$

or

$$W(\eta(r), \vartheta) \cap R(x_0, r) \subset M^*(\eta(r)) = \{x: \{\eta(r), x\} \in M\}$$

3) the system of sets $W(t, \vartheta)$ $(t_0 \leqslant t \leqslant \eta(r))$ is u-stable in M.

Proof. Let τ_* be the upper bound of all τ satisfying

$$\tau \leqslant \vartheta, \quad W(t, \vartheta) \cap R(x_0, r) \neq \phi \quad \text{for } t \in [t_0, \tau]$$

The set of these numbers τ is nonempty, since $\tau = t_0$ satisfies the above requirements. We shall show that τ_* may be taken as $\eta(r)$, i.e. the number τ_* satisfies the Conditions (1) – (3).

First it is necessary to confirm that Condition (1) holds. For this purpose it is sufficient to show that the set $W(\tau_*, \vartheta) \cap R(x_0, r)$ is nonempty. Let τ_k $(k = 1, 2, \ldots)$ be a sequence of numbers converging to τ_* from the left. Every set $W(\tau_k, \vartheta) \cap R(x_0, r)$ $(k = 1, 2, \ldots)$ is nonempty and the sets are all equally restricted, therefore a sequence of points w_k, $w_k \in W(\tau_k, \vartheta) \cap R(x_0, r)$ $(k = 1, 2, \ldots)$ can be selected which will converge to some point w_*.

Obviously $w_* \in R(x_0, r)$, it will be shown that $w_* \in W(\tau_*, \vartheta)$. Assume the opposite is true, i.e. $w_* \notin W(\tau_*, \vartheta)$. Then by Lemma 3.2 it follows that $\varepsilon > 0$ exists such that

$$R(w_*, \varepsilon) \cap W(\tau_*, \vartheta) = \phi \tag{3.24}$$

holds.

Since the points $p_k = \{\tau_k, w_k\}$ belong to the set D (see (3.23)), the closure of D implies that

$$\{\tau_*, w_*\} > D \tag{3.25}$$

From (3.24),(3.25) and (3.23) it follows that the point $p_* = \{\tau_*, w_*\}$ cannot belong to the set M. Therefore the closure of M implies that

$$\rho(\{t_*, w_*\}, M) > 0 \tag{3.26}$$

Let $\nu(dv)$ be some measure on Q. From (3.24) and (3.26) it follows that for sufficiently large k the conditions

$$\rho(\{t, x[t]\}, M) > 0 \quad \text{for} \quad t \in [\tau_k, \tau_*]$$

$$x[\tau_*] \in R(w_*, \varepsilon), \quad x[\tau_*] \notin W(\tau_*, \vartheta)$$

hold for any motion $x[t] = x[t; \tau_k, w_k, U_\tau, \nu(dv)]$. Therefore, by Lemma 3.1, the relation $w_k \notin W(\tau_k, \vartheta)$ holds for sufficiently large k. The contradiction obtained proves the validity of the inclusion

$$w_* \in W(\tau_*, \vartheta) \cap R(x_0, r)$$

Consequently Condition (1) holds for the number $\tau_* = \eta(r)$.

We shall now verify that Condition (2) is satisfied. Again assume the opposite, i.e. suppose that a point $w_* \in W(\tau_*, \vartheta) \cap R(x_0, r)$ exists for which the relations

$$\| w_* - x_0 \| < r, \qquad w_* \notin M(\tau_*)$$

hold simultaneously and the second of these includes the case when the set $M(\tau_*)$ is empty. From the definition of the number τ_* a sequence of numbers t_k converging to τ_* from the right exists such that the sets $W(t_k, \vartheta) \cap R(x_0, r)$ $(k = 1, 2, \ldots)$ are empty.

Thus the following relations may be written

$$t_k > \tau_*, \quad \lim_{k \to \infty} t_k = \tau_*, \quad W(t_k, \vartheta) \cap R(x_0, r) = \phi$$

$$\| w_* - x_0 \| < r, \quad \rho(\{\tau_*, w_*\}, M) > 0$$

Let $\nu\ (dv)$ be some measure on Q . Then the above relations imply that for sufficiently large k the conditions $\rho\ (\{t,\ x\ [t]\},\ M) > 0 \quad \text{for } t \in [\tau_*,\ t_k]$

$$\|x\ [t_k] - x_0\| \leqslant r, \qquad \text{i.e. } x\ [t_k] \notin W\ (t_k,\ \vartheta)$$

hold for any motion $x\ [t] = x\ [t;\ \tau_*,\ w_*,\ U_\tau,\ \nu\ (dv)]$.

Consequently, by Lemma 3.1, $w_* \notin W\ (\tau_*,\ \vartheta)$ which contradicts the assumption. This contradiction proves that Condition (2) is satisfied for $\tau_* = \eta\ (r)$.

This leaves only Condition (3) to be proved. Again assume that it does not hold, i.e. that numbers $t_* \in [t_0,\ \tau_*)$, $\delta \in (0,\ \tau_* - t_*]$, a point $w_* \in W\ (t_*,\ \vartheta)$ and some measure $\nu_*\ (dv)$ exist such that the relations

$$\rho\ (\{t,\ x\ [t]\},\ M) > 0 \quad \text{for } t_* \leqslant t \leqslant t_* + \delta \qquad (3.27)$$
$$x\ [t_* + \delta] \notin W\ (t_* + \delta,\ \vartheta)$$

hold for any motion $x\ [t] = x\ [t;\ t_*,\ w_*,\ U_\tau,\ \nu_*\ (dv)]$. By Lemma 3.1, from (3.27) it follows at once that $w_* \notin W\ (t_*,\ \vartheta)$. This contradiction proves that Condition (3) holds.

Setting $\eta\ (r) = \tau_*$ we find that τ_* satisfies Conditions (1) – (3) formulated in Lemma 3.3, thus proving the latter.

Lemmas 3.2 and 3.3 may now be used to prove the following theorem, from which the alternative stated at the end of Sect. 1 immediately follows.

Theorem 3.1. If $x_0 \in W\ (t_0,\ \vartheta)$, then the mixed strategy of the first player $U^{(e)} = U^{(e)}\ (t,\ x)$ extremal to the system of sets $W\ (t,\ \vartheta)\ \ (t_0 \leqslant t \leqslant \vartheta)$ guarantees that the conditions

$$\vartheta\ (x\ [\cdot];\ M) \leqslant \vartheta, \quad \{t,\ x\ [t]\} \in D \quad \text{for } t \in [t_0,\ \vartheta\ (x\ [\cdot];\ M)]$$

are fulfilled for any motion $x\ [t] = x\ [t;\ t_0,\ x_0,\ U^{(e)},\ V_\tau]$.

If on the other hand $x_0 \notin W(t_0,\ \vartheta)$, then such a positive number $\varepsilon > 0$ and such a mixed strategy of the second player $V_* = V_*\ (t,\ x)$ both exist, that the condition

$$\{t,\ x\ [t]\} \notin M^\varepsilon \qquad \text{for} \qquad t_0 \leqslant t \leqslant \min\ \{\vartheta,\ \tau^\varepsilon\ (x\ [\cdot];\ D)\}$$

will hold for any motion $x\ [t] = x\ [t;\ t_0,\ x_0,\ U_\tau,\ V_*]$.

Here $\tau^\varepsilon\ (x\ [\cdot];\ D)$ denotes the instant at which the relation $\rho\ (\{t,\ x\ [t]\},\ D) = \varepsilon$ first becomes valid.

Proof. Let $x_0 \in W\ (t_0,\ \vartheta)$ and let a sufficiently large number r be chosen such that the condition $\max_t \|x\ [t] - x_0\| < r \quad \text{for } t_0 \leqslant t \leqslant \vartheta \qquad (3.28)$

holds for any motion $x\ [t] = x\ [t;\ t_0,\ x_0,\ U_\tau,\ V_\tau]$.

By Lemma 3.3 we can find a number $\eta\ (r) \in [t_0,\ \vartheta]$ corresponding to r which will satisfy Conditions (1) – (3) of this Lemma.

Now consider a system of sets $W\ (t,\ \vartheta)\ (t_0 \leqslant t \leqslant \eta\ (r))$ which, by virtue of Lemmas 3.2 and 3.3, satisfies all the conditions of Lemma 2.1. By Lemma 2.1, the relation

$$x\ [t] \in W\ (t,\vartheta) \quad \text{for } t_0 \leqslant t \leqslant \min\ \{\eta\ (r),\ \vartheta\ (x\ [\cdot];\ M)\} \qquad (3.29)$$

holds for any motion $x\ [t] = x\ [t;\ t_0,\ x_0,\ U^{(e)},\ V_\tau]$.

We shall show that $\vartheta\ (x\ [\cdot];\ M) \leqslant \eta\ (r)$. Indeed, if $\rho\ (x_0,\ W\ (\eta\ (r),\ \vartheta)) = r$ (see Condition (2) of Lemma 3.3), by (3.28) and (3.29) the following strict inequality $\vartheta\ (x\ [\cdot];\ M) < \eta\ (r)$ will hold; if, on the other hand, the relations

$$W\ (\eta\ (r),\ \vartheta) \cap R\ (x_0,\ r) \subset M^*\cdot(\eta\ (r)) \text{ and } \min\ \{\vartheta\ (x\ [\cdot];\ M),\ \eta\ (r)\} = \eta\ (r)$$

are valid, then by (3.28) and (3.29) the inclusion

$$x[\eta(r)] \in W(\eta(r), \vartheta) \cap R(x_0, r) \subset M^*(\eta(r))$$

follows, i.e. in this case $\vartheta(x[\cdot]; M) = \eta(r)$. It follows, therefore, that the inequality $\vartheta(x[\cdot]; M) \leqslant \eta(r) \leqslant \vartheta$ holds for any motion $x[t] = x[t; t_0, x_0, U^{(e)}, V_\tau]$. The phase restriction that $\{t, x[t]\} \in D$ for all $t \in [t_0, \vartheta(x[\cdot]; M)]$ now follows from (3.29) and from the inclusion $W(t, \vartheta) \subset D^*(t)$ (see Lemma 3.2). This proves the first statement of Theorem 3.1).

Let now $x_0 \notin W(t_0, \vartheta)$. The definition of the set $W(t, \vartheta)$ now implies that a mixed strategy of the second player $V_* = V_*(t, x)$ exists such that the inequality

$$\varkappa(x[\cdot]; t_0, \vartheta) > 0 \tag{3.30}$$

holds for any motion $x[t] = x[t; t_0, x_0, U_\tau, V_*]$. Since the set of motions $x[t; t_0, x_0, U_\tau, V_*]$ is compact in itself, (3.30) implies that a positive number $\varepsilon > 0$ exists such that the relation $\{t, x[t]\} \notin M^\varepsilon$ for all $t_0 \leqslant t \leqslant \min\{\vartheta, \tau^\varepsilon(x[\cdot]; D)\}$ holds for any motion $x[t] = x[t; t_0, x_0, U_\tau, V_*]$. This proves the second statement of Theorem 3.1.

To conclude this Section some statements are given which might be of some use in the study of differential games.

Let $W_*(\tau, \vartheta)$ $(t_0 \leqslant \tau \leqslant \vartheta)$ be a set of points w satisfying the condition that for any mixed strategy of the second player $V = V(t, x)$ a motion

$$x[t] = x[t; \tau, w, U_\tau, V]$$

exists for which the inclusion

$$\{t, x[t]\} \in D \quad \text{for} \quad \tau \leqslant t \leqslant \min\{\vartheta, \vartheta(x[\cdot]; M)\}$$

holds. Here, as before, $\vartheta(x[\cdot]; M)$ denotes the instant at which the point $p[t] = \{t, x[t]\}$ reaches the set M for the first time.

T h e o r e m 3.2. If $x_0 \in W_*(t_0, \vartheta)$, then the mixed strategy of the first player $U^{(e)} = U^{(e)}(t, x)$ extremal to the system of sets $W_*(t, \vartheta)$ $(t_0 \leqslant t \leqslant \vartheta)$ ensures that the condition

$$\{t, x[t]\} \in D \qquad \text{for} \quad t_0 \leqslant t \leqslant \min\{\vartheta, \vartheta(x[\cdot]; M)\}$$

holds for any motion

$$x[t] = x[t; t_0, x_0, U^{(e)}, V_\tau]$$

If $x_0 \notin W_*(t, \vartheta)$, then a mixed strategy of the second player $V_* = V_*(t, x)$ and a number $\varepsilon > 0$ exist such that the conditions

$$\tau^\varepsilon(x[\cdot]; D) \leqslant \vartheta, \quad \rho(\{t, x[t]\}, M) > \varepsilon \quad \text{for} \quad t_0 \leqslant t \leqslant \tau^\varepsilon(x[\cdot]; D)$$

hold for any motion $x[t] = x[t; t_0, x_0, U_\tau, V_*]$.

P r o o f. Let M_* be the union of the set M and a hyperplane $t = \vartheta = \text{const}$. Then from the definition of the sets $W_*(t, \vartheta)$ it follows that $W_*(t, \vartheta)$ is an aggregate of all points $\{w\}$ such that, from the position $p = \{t, w\}$, the set M_* is positionally absorbed by D at the instant ϑ. It can easily be now noted that the correctness of Theorem 3.2 follows from Theorem 3.1 and from the definition of M_*.

N o t e 3.1. It is easily seen that the statement of Theorem 3.1 remains valid when the positions of the players in its formulation and in the definition of $W(\tau, \vartheta)$ are interchanged, i.e. the set $W(\tau, \vartheta)$ is defined as an aggregate of points w for which (3.2) holds for at least one of the motions $x[t] = x[t; \tau, w, U, V_\tau]$, irrespective of the choice of the strategy U of the first player; moreover, the strategies $U^{(e)}$ and U_τ of the first player in the formulation of Theorem 3.1 should be replaced by the strategies

$V^{(\varepsilon)}$ and V_τ of the second player, and the strategies V_τ and V^* of the second player replaced by the strategies U_τ and U^* of the first player. This is equally applicable to Theorem 3.2.

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THE ACCURACY OF CERTAIN NONLINEAR CONTROL SYSTEMS WITH RESTRICTIONS AND LAG

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The accuracy with which a nonlinear control system with lag reproduces an arbitrary action belonging to a certain class of functions is examined. The maximum errors arising in reproducing the action and their dependence on the parameters of the controlled object, and on the law of control used, are estimated.

1. Statement of the problem. Consider a closed system consisting of a controlled object and a regulator. The purpose of this system is to reproduce, using the initial value of the object $y(t)$, a previously unknown controlling action $x(t)$ whose rate of change

$$x'(t) \equiv \varphi(t), \quad |\varphi(t)| \leqslant m, \quad x(0) = 0 \tag{1.1}$$

is bounded, belonging to the class of functions F. The quality of performance of the system, which is at rest at $t \leqslant 0$, will be characterized by the maximum error

$$\varepsilon_{\max}(t) = \max|\varepsilon(t)|, \quad \varepsilon(t) = x(t) - y(t) \qquad (x \in F) \tag{1.2}$$